

The Putnam Competition is the premier national undergraduate mathematics contest, which will next be held on Saturday, December 2, 2017.\* Approximately 4,200 undergraduate students from 570 colleges and universities throughout the U.S. and Canada are expected to compete. Registration is free. In addition to awarding cash prizes (up to \$3,500) to the 25 highest scoring participants, the Mathematical Association of America will publish the names and schools of the top 500 students. The national directors of the Putnam Competition make a point of commending these students to the attention of graduate school admissions committees. (There is no risk in attempting the Putnam: participants scoring below the top 500 are not publicly identified.)

In 2016 a score of 29/120 would have put you in the top 500. The exam, needless to say, is not easy! *If you can solve even one problem completely correctly, you will be making a significant contribution to the SLU team.* A good performance can bring great prestige to both you and the university. The Putnam provides an opportunity for the creative application to novel problems of important mathematical techniques and ideas spanning much of the undergraduate curriculum.

Interested students should contact the local supervisor, Greg Marks ([marks@member.ams.org](mailto:marks@member.ams.org)), and might consider enrolling in Professor Marks's one-unit S/U course Math 2690, Mathematical Problem Solving.

**HISTORY.** The competition began in 1938 and is designed to stimulate a healthful rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the Harvard Graduates' Magazine in which he described the merits of an intellectual intercollegiate competition.

**DESCRIPTION.** The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as "survey of the foundations of mathematics."

## Sample Exam

### The Seventy-Seventh William Lowell Putnam Mathematical Competition

#### Saturday, December 3, 2016

*Problems A1 through A6 were given during the 3-hour morning session of the exam; problems B1 through B6 were given during the 3-hour afternoon session.*

- A1.** Find the smallest positive integer  $j$  such that for every polynomial  $p(x)$  with integer coefficients and for every integer  $k$ , the integer

$$p^{(j)}(k) = \left. \frac{d^j}{dx^j} p(x) \right|_{x=k}$$

(the  $j$ -th derivative of  $p(x)$  at  $k$ ) is divisible by 2016.

- A2.** Given a positive integer  $n$ , let  $M(n)$  be the largest integer  $m$  such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n \rightarrow \infty} \frac{M(n)}{n}.$$

- A3.** Suppose that  $f$  is a function from  $\mathbb{R}$  to  $\mathbb{R}$  such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

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\*The registration deadline is October 1, 2017.

for all real  $x \neq 0$ . (As usual,  $y = \arctan x$  means  $-\pi/2 < y < \pi/2$  and  $\tan y = x$ .) Find

$$\int_0^1 f(x) dx.$$

- A4.** Consider a  $(2m - 1) \times (2n - 1)$  rectangular region, where  $m$  and  $n$  are integers such that  $m, n \geq 4$ . This region is to be tiled using tiles of the two types shown:



(The dotted lines divide the tiles into  $1 \times 1$  squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?

- A5.** Suppose that  $G$  is a finite group generated by the two elements  $g$  and  $h$ , where the order of  $g$  is odd. Show that every element of  $G$  can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \dots g^{m_r} h^{n_r}$$

with  $1 \leq r \leq |G|$  and  $m_1, n_1, m_2, n_2, \dots, m_r, n_r \in \{1, -1\}$ . (Here  $|G|$  is the number of elements of  $G$ .)

- A6.** Find the smallest constant  $C$  such that for every real polynomial  $P(x)$  of degree 3 that has a root in the interval  $[0, 1]$ ,

$$\int_0^1 |P(x)| dx \leq C \max_{x \in [0, 1]} |P(x)|.$$

- B1.** Let  $x_0, x_1, x_2, \dots$  be the sequence such that  $x_0 = 1$  and for  $n \geq 0$ ,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function  $\ln$  is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \dots$$

converges and find its sum.

- B2.** Define a positive integer  $n$  to be *squarish* if either  $n$  is itself a perfect square or the distance from  $n$  to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is  $45^2 = 2025$  and  $2025 - 2016 = 9$  is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer  $N$ , let  $S(N)$  be the number of squarish integers between 1 and  $N$ , inclusive. Find positive constants  $\alpha$  and  $\beta$  such that

$$\lim_{N \rightarrow \infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

- B3.** Suppose that  $S$  is a finite set of points in the plane such that the area of triangle  $\triangle ABC$  is at most 1 whenever  $A, B$ , and  $C$  are in  $S$ . Show that there exists a triangle of area 4 that (together with its interior) covers the set  $S$ .

- B4.** Let  $A$  be a  $2n \times 2n$  matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability  $1/2$ . Find the expected value of  $\det(A - A^t)$  (as a function of  $n$ ), where  $A^t$  is the transpose of  $A$ .

- B5.** Find all functions  $f$  from the interval  $(1, \infty)$  to  $(1, \infty)$  with the following property:

$$\text{if } x, y \in (1, \infty) \text{ and } x^2 \leq y \leq x^3, \text{ then } (f(x))^2 \leq f(y) \leq (f(x))^3.$$

- B6.** Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k 2^n + 1}.$$