Solutions: Final Exam

1. (13 pts.) No partial credit is available on this problem, so be meticulous. Write down the set of all solutions $(x, y, z) \in \mathbb{R}^3$ to the following system of equations:

$$\begin{array}{rcrcrcr} 2x & - & y & - & 3z & = & -6 \\ 3x & - & 8y & - & 2z & = & 10 \\ -x & + & 2y & + & z & = & -1 \end{array} \tag{1}$$

Solution. Here is one approach. Add 2 times the third row to the second row and 3 times the third row to the first row; then add the second row to the first row:

$$(1) \quad \Longleftrightarrow \quad \begin{array}{rcrcrcr} -x & + & 5y & & & = & -9 \\ x & - & 4y & & & = & 8 \\ -x & + & 2y & + & z & = & -1 \end{array} \quad \Longleftrightarrow \quad \begin{array}{rcrcrcr} & & y & & & = & -1 \\ x & - & 4y & & & = & 8 \\ -x & + & 2y & + & z & = & -1 \end{array}$$

Finish by back-substitution. The first equation is $y = -1$, whence the second equation yields $x = 4y + 8 = 4$, whence the third equation yields $z = x - 2y - 1 = 5$. The system (1) has a unique solution: $(x, y, z) = (4, -1, 5)$.

2. (a) (9 pts.) Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of some vector space $\mathcal{V}$. By $\mathcal{S} + \mathcal{T}$ we mean the set of all vectors of the form $\mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in \mathcal{S}$ and $\mathbf{y} \in \mathcal{T}$. Prove that $\mathcal{S} + \mathcal{T}$ is a subspace of $\mathcal{V}$.

Solution. Suppose $\mathbf{u}, \mathbf{v} \in \mathcal{S} + \mathcal{T}$ and $c \in \mathbb{R}$. By definition of $\mathcal{S} + \mathcal{T}$, we can write $\mathbf{u} = \mathbf{s}_1 + \mathbf{t}_1$ and $\mathbf{v} = \mathbf{s}_2 + \mathbf{t}_2$ for some vectors $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}$ and $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}$. Since $\mathcal{S}$ and $\mathcal{T}$ are subspaces, we have $c\mathbf{s}_1 \in \mathcal{S}$, $\mathbf{s}_1 + \mathbf{s}_2 \in \mathcal{S}$, $c\mathbf{t}_1 \in \mathcal{T}$, and $\mathbf{t}_1 + \mathbf{t}_2 \in \mathcal{T}$. Therefore $\mathbf{u} + \mathbf{v} = (\mathbf{s}_1 + \mathbf{s}_2) + (\mathbf{t}_1 + \mathbf{t}_2) \in \mathcal{S} + \mathcal{T}$ and $c\mathbf{u} = c\mathbf{s}_1 + c\mathbf{t}_1 \in \mathcal{S} + \mathcal{T}$, which proves that $\mathcal{S} + \mathcal{T}$ is a subspace of $\mathcal{V}$.

(b) (9 pts.) Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of some vector space $\mathcal{V}$. Suppose that $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ is a set of vectors in $\mathcal{S}$ that are linearly independent, and that $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$ is a set of vectors in $\mathcal{T}$ that are linearly independent. Prove that if $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$ then $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$ is a linearly independent set of vectors.

Solution. Suppose

$$c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_k\mathbf{s}_k + d_1\mathbf{t}_1 + d_2\mathbf{t}_2 + \cdots + d_\ell\mathbf{t}_\ell = \mathbf{0} \tag{2}$$

for some scalars $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_\ell$. Then

$$c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_k\mathbf{s}_k = -(d_1\mathbf{t}_1 + d_2\mathbf{t}_2 + \cdots + d_\ell\mathbf{t}_\ell) \in \mathcal{S} \cap \mathcal{T} \qquad \text{and} \qquad \mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$$

imply that

$$c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_k\mathbf{s}_k = \mathbf{0} = -(d_1\mathbf{t}_1 + d_2\mathbf{t}_2 + \cdots + d_\ell\mathbf{t}_\ell). \tag{3}$$

By linear independence of $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ and $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$, we infer from Equation (3) that every c_i and every d_i equals 0.

Since Equation (2) implies $c_1 = c_2 = \cdots = c_k = d_1 = d_2 = \cdots = d_\ell = 0$, the vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$ are linearly independent.

3. (18 pts.) Let $\mathcal{V}$ be an inner product space with inner product $(\ ,\ ) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$. Fix a vector $\mathbf{v} \in \mathcal{V}$. (Do not make any additional assumptions about what $\mathcal{V}$ or $(\ ,\ )$ or $\mathbf{v}$ is.)

(a) Decide whether the quoted statement is true or false: "The function $f : \mathcal{V} \longrightarrow \mathbb{R}$ defined by

$$f(\mathbf{u}) = (\mathbf{u}, \mathbf{v}) \qquad \textit{for each } \mathbf{u} \in \mathcal{V}$$

is a linear transformation from $\mathcal{V}$ to $\mathbb{R}$." (You must prove your answer to receive any credit.)

Solution. The statement is true. Suppose $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{V}$ and $c \in \mathbb{R}$. By the additivity axiom of inner products, $(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) = (\mathbf{u}_1, \mathbf{v}) + (\mathbf{u}_2, \mathbf{v})$. By the homogeneity axiom, $(c\mathbf{u}_1, \mathbf{v}) = c(\mathbf{u}_1, \mathbf{v})$. Hence

$$f(\mathbf{u}_1 + \mathbf{u}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) = (\mathbf{u}_1, \mathbf{v}) + (\mathbf{u}_2, \mathbf{v}) = f(\mathbf{u}_1) + f(\mathbf{u}_2)$$

and

$$f(c\mathbf{u}_1) = (c\mathbf{u}_1, \mathbf{v}) = c(\mathbf{u}_1, \mathbf{v}) = c\, f(\mathbf{u}_1),$$

proving that f is a linear transformation.

(b) Decide whether the quoted statement is true or false: "The set

$$\{\mathbf{u} \in \mathcal{V} \;:\; (\mathbf{u}, \mathbf{v}) = 0\}$$

is a subspace of $\mathcal{V}$." (You must prove your answer to receive any credit.)

Solution. The statement is true, since the given set is the kernel of the function $f : \mathcal{V} \longrightarrow \mathbb{R}$ defined in part (a), and the kernel of any linear transformation is a subspace.

4. (13 pts.) Show that the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for $\mathbb{R}^4$.

Solution. Since the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

has determinant 6 (most easily seen by expanding along the fourth column), and any matrix with nonzero determinant has linearly independent columns, the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ are linearly independent. Four linearly independent vectors in a 4-dimensional vector space form a basis.

5. (18 pts.) Let

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 6 & 4 & -9 \\ 4 & 4 & -7 \end{bmatrix}.$$

(a) Find all eigenvalues and eigenvectors of A.

Solution. We first compute all eigenvalues by finding the roots of the characteristic polynomial of A:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 3 & -3 \\ 6 & 4-\lambda & -9 \\ 4 & 4 & -7-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 2+\lambda & -3 \\ 6 & -2-\lambda & -9 \\ 4 & 0 & -7-\lambda \end{vmatrix} && \text{(subtracting the first column from the second column)} \\
&= \begin{vmatrix} 1-\lambda & 2+\lambda & -3 \\ 7-\lambda & 0 & -12 \\ 4 & 0 & -7-\lambda \end{vmatrix} && \text{(adding the first row to the second row)} \\
&= -(2+\lambda) \begin{vmatrix} 7-\lambda & -12 \\ 4 & -7-\lambda \end{vmatrix} && \text{(expanding the determinant along the second column)} \\
&= -(2+\lambda)(\lambda^2 - 1) \\
&= -(\lambda+2)(\lambda+1)(\lambda-1).
\end{aligned}$$

The eigenvalues are the roots of this polynomial: $\lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 1$.

Now we will find eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ corresponding to the eigenvalues λ_1, λ_2, and λ_3. This is done by finding a basis for the nullspace of the matrix $A - \lambda_i I$ for each $i = 1, 2, 3$. Remember that performing elementary row operations on a matrix does not alter its nullspace.

An eigenvector $\mathbf{v}_1$ corresponding to the eigenvalue $\lambda_1 = -2$ is a basis for the nullspace

$$\begin{aligned} \text{NS}\begin{bmatrix} 1-\lambda_1 & 3 & -3 \\ 6 & 4-\lambda_1 & -9 \\ 4 & 4 & -7-\lambda_1 \end{bmatrix} &= \text{NS}\begin{bmatrix} 3 & 3 & -3 \\ 6 & 6 & -9 \\ 4 & 4 & -5 \end{bmatrix} = \text{NS}\begin{bmatrix} 3 & 3 & -3 \\ 6 & 6 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \text{NS}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{NS}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{NS}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}. \end{aligned}$$

An eigenvector $\mathbf{v}_2$ corresponding to the eigenvalue $\lambda_2 = -1$ is a basis for the nullspace

$$\begin{aligned} \text{NS}\begin{bmatrix} 1-\lambda_2 & 3 & -3 \\ 6 & 4-\lambda_2 & -9 \\ 4 & 4 & -7-\lambda_2 \end{bmatrix} &= \text{NS}\begin{bmatrix} 2 & 3 & -3 \\ 6 & 5 & -9 \\ 4 & 4 & -6 \end{bmatrix} = \text{NS}\begin{bmatrix} 2 & 3 & -3 \\ 6 & 5 & -9 \\ 0 & 0 & 0 \end{bmatrix} = \text{NS}\begin{bmatrix} 2 & 3 & -3 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{NS}\begin{bmatrix} 2 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{NS}\begin{bmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right\}. \end{aligned}$$

An eigenvector $\mathbf{v}_3$ corresponding to the eigenvalue $\lambda_3 = 1$ is a basis for the nullspace

$$\begin{aligned} \text{NS}\begin{bmatrix} 1-\lambda_3 & 3 & -3 \\ 6 & 4-\lambda_3 & -9 \\ 4 & 4 & -7-\lambda_3 \end{bmatrix} &= \text{NS}\begin{bmatrix} 0 & 3 & -3 \\ 6 & 3 & -9 \\ 4 & 4 & -8 \end{bmatrix} = \text{NS}\begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & -3 \\ 1 & 1 & -2 \end{bmatrix} = \text{NS}\begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{NS}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{NS}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

In summary, the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -1$, and $\lambda_3 = 1$; corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(of course, any nonzero scalar multiple of $\mathbf{v}_i$ is also an eigenvector with eigenvalue λ_i).

(b) Find an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

Solution. The columns of Q are the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in part (a); the diagonal entries of D are the eigenvalues λ_1, λ_2, and λ_3. Thus,

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1}.$$

(c) Compute A^n where $n = 10^{100}$.

Solution. Since $A = QDQ^{-1}$ implies $A^n = QD^nQ^{-1}$, by part (b) we have

$$\begin{aligned}
A^{10^{100}} &= \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10^{100}} \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10^{100}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10^{100}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ -1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} \quad = \quad \begin{bmatrix} 2^{1+10^{100}} - 1 & 2^{10^{100}} - 1 & -3 \cdot 2^{10^{100}} + 3 \\ -2^{1+10^{100}} + 2 & -2^{10^{100}} + 2 & 3 \cdot 2^{10^{100}} - 3 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

6. (20 pts.) A matrix A is called idempotent if $A = A^2$.

(a) Give an example of an idempotent 2 by 2 matrix whose four entries are all nonzero.

Solution #1. Since a matrix similar to an idempotent matrix is clearly idempotent, one example is the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution #2. Another example is any projection matrix $P = A(A^tA)^{-1}A^t$ that gives the orthogonal projection onto a line in $\mathbb{R}^2$ that is not horizontal or vertical:

$$P = \begin{bmatrix} a \\ b \end{bmatrix} \left(\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)^{-1} \begin{bmatrix} a & b \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} \dfrac{a^2}{a^2+b^2} & \dfrac{ab}{a^2+b^2} \\ \dfrac{ab}{a^2+b^2} & \dfrac{b^2}{a^2+b^2} \end{bmatrix}$$

where a and b are nonzero.

(b) What can you say about an invertible idempotent matrix?

Solution. It must be the identity matrix I. If A^{-1} exists, and $A^2 = A$, then $A^{-1}A^2 = A^{-1}A$, i.e. $A = I$.

(c) What are the possible eigenvalues of an idempotent matrix?

Solution. Zero and one. If $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$, then since $A = A^2$ we have

$$\lambda\mathbf{v} = A\mathbf{v} = A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}.$$

Thus, $(\lambda^2 - \lambda)\mathbf{v} = \mathbf{0}$, and since $\mathbf{v} \neq \mathbf{0}$ we see that $\lambda^2 - \lambda = 0$, which means that $\lambda = 0$ or $\lambda = 1$. (Note that both possibilities can occur: see the examples given in part (a).)

(d) Decide whether the quoted statement is true or false: "If A and B are idempotent matrices of the same size, then AB is an idempotent matrix." (You must prove your answer to receive any credit.)

Solution. The statement is false. One counterexample is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(e) Decide whether the quoted statement is true or false: "Every idempotent matrix is diagonalizable." (You must prove your answer to receive any credit.)

Solution. The statement is true. If A is an n by n idempotent matrix, then $A^2 = A$ implies that $A\mathbf{v} = \mathbf{v}$ whenever $\mathbf{v}$ is a column vector of A. So the entire column space of A is contained in the eigenspace corresponding to the eigenvalue 1. The nullspace of A is the eigenspace corresponding to the eigenvalue 0. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for the column space of A; let $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ be a basis for the nullspace of A. By the Rank-Nullity Theorem, $k+\ell = n$. Because eigenspaces for distinct eigenvalues of a matrix have intersection $\{\mathbf{0}\}$,* Problem 2(b) implies that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ are linearly independent. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ is a set of n linearly independent eigenvectors of A, which shows that A is diagonalizable.

7. (Extra Credit) As usual, let $\mathcal{C}[0,\,1]$ denote the real vector space consisting of all continuous functions from the closed interval $[0,\,1]$ to $\mathbb{R}$. Give an example of a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ with the property that $\dim(\ker T) = 2$.

Solution #1. More generally, for any integer $n \geqslant 2$ we will exhibit a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ with $\dim(\ker T) = n$. Given $f \in \mathcal{C}[0,\,1]$, define $T(f) \in \mathcal{C}[0,\,1]$ by

$$T(f)(x) = f(x) - g(x) \qquad \text{where} \qquad g(x) = \sum_{k=0}^{n-1} \left(f\left(\frac{k}{n-1}\right) \cdot \frac{\displaystyle\prod_{\substack{0\leqslant m\leqslant n-1\\ m\neq k}} \left(x - \frac{m}{n-1}\right)}{\displaystyle\prod_{\substack{0\leqslant m\leqslant n-1\\ m\neq k}} \left(\frac{k}{n-1} - \frac{m}{n-1}\right)} \right).$$

Note that $g(x)$ is given by the Lagrange interpolation formula for the unique polynomial of degree less than n that agrees with $f(x)$ at the n different values $x = 0,\ 1/(n-1),\ 2/(n-1),\ 3/(n-1),\ \ldots,\ 1$.

Consequently, if f is a polynomial of degree less than n, then $g = f$, and so $f \in \ker T$. Thus, the set of all polynomials of degree less than n is a subset of $\ker T$.

On the other hand, if $f \in \ker T$ then $f(x) = g(x)$ for all $x \in [0,\,1]$, which means that $f \in \mathcal{C}[0,\,1]$ is a polynomial of degree less than n (since g is). Therefore, $\ker T$ is a subset of the set of all polynomials of degree less than n.

We have shown that $\ker T$ equals the set of all polynomials of degree less than n, which is an n-dimensional vector space.†

Specializing to the case $n = 2$, the formula for the linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ becomes‡

$$T(f)(x) = f(x) - \Big(\Big[f(1) - f(0)\Big]x + f(0)\Big),$$

and $T(f) = \mathbf{0}$ if and only if $f(x) = ax + b$ for some constants $a, b \in \mathbb{R}$. Thus, $\ker T$ equals the set of all such functions, a vector space of dimension 2.

Solution #2. Still more generally, we can define a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ whose kernel has any prescribed dimension $\alpha \leqslant |\dim(\mathcal{C}[0,\,1])|$ (where we use $|X|$ to denote the cardinality of the set X). In fact, we can even define a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ whose kernel equals any prescribed subspace $\mathcal{S} \subseteq \mathcal{C}[0,\,1]$. Of course, $\mathcal{C}[0,\,1]$ contains subspaces of every dimension up to $|\dim(\mathcal{C}[0,\,1])|$ (indeed, given $\alpha \leqslant |\dim(\mathcal{C}[0,\,1])|$, choose a basis§ for $\mathcal{C}[0,\,1]$, pick any subset of this basis of cardinality α, and define $\mathcal{S}$ to be the span of this subset of basis vectors).

Given any subspace $\mathcal{S} \subseteq \mathcal{C}[0,\,1]$, we construct a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ with $\ker T = \mathcal{S}$ as follows. Pick a basis $\{\mathbf{s}_i\}_{i\in I}$ for $\mathcal{S}$. Since any linearly independent set in a vector space can be enlarged to a basis, there exists a basis for $\mathcal{C}[0,\,1]$ of the form $\{\mathbf{s}_i\}_{i\in I} \cup \{\mathbf{v}_j\}_{j\in J}$ (so for every $j \in J$ we have $\mathbf{v}_j \notin \mathcal{S}$). Now define a linear transformation $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ by

$$T(\mathbf{s}_i) = \mathbf{0} \quad \text{for every } i \in I \qquad \text{and} \qquad T(\mathbf{v}_j) = \mathbf{v}_j \quad \text{for every } j \in J$$

(recall that a linear transformation is fully determined by what it does to a basis).¶ It is easy to verify that $\ker T = \mathcal{S}$.

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* *Not* because the column space and nullspace of a matrix have intersection $\{\mathbf{0}\}$! That is false for matrices in general. (The row space and nullspace of any matrix do have intersection $\{\mathbf{0}\}$, since the row space and nullspace are orthogonal.)

† This construction is also possible in the case where $n = 1$; we can define $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ by $T(f)(x) = f(x) - f(0)$. And if we want $\dim(\ker T) = 0$, let T be the identity transformation.

‡ Notice that $y = \Big[f(1) - f(0)\Big]x + f(0)$ is the equation of the line segment connecting the endpoints of the graph of f.

§ By Zorn's Lemma, every vector space has a basis.

¶ In fact, we could let T map the $\mathbf{v}_j$'s to any linearly independent set of vectors in $\mathcal{C}[0,\,1]$ (e.g. T could permute the $\mathbf{v}_j$'s arbitrarily). Thus, whenever J is infinite (e.g. whenever $\mathcal{S}$ is finite-dimensional), there exist uncountably many different linear transformations $T : \mathcal{C}[0,\,1] \longrightarrow \mathcal{C}[0,\,1]$ with kernel $\mathcal{S}$.

As a "concrete" example of a solution to the given problem, start with your favorite two (linearly independent) continuous functions. Say, take f to be the function in $\mathcal{C}[0, 1]$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$, and take g to be any function in $\mathcal{C}[0, 1]$ with the property that for every $x \in [0, 1]$ the derivative $g'(x)$ does not exist.[‖] The construction of the previous paragraph yields a linear transformation $T : \mathcal{C}[0, 1] \longrightarrow \mathcal{C}[0, 1]$ whose kernel is the subspace $\{c_1 f + c_2 g \; : \; c_1, c_2 \in \mathbb{R}\} \subseteq \mathcal{C}[0, 1]$, which is 2-dimensional.

. . . .

Remarks

Problem 2(a) was Exercise 32 of §1.5, on p. 72 of Penney (assigned as part of Problem Set 5). If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathcal{V}$, their intersection $\mathcal{S} \cap \mathcal{T}$ is always a subspace of $\mathcal{V}$; however, their union $\mathcal{S} \cup \mathcal{T}$ is not a subspace in general. Note that $\mathcal{S} + \mathcal{T}$ is the smallest subspace of $\mathcal{V}$ that contains $\mathcal{S} \cup \mathcal{T}$.

In Problem 2(b), if the vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ span $\mathcal{S}$, and the vectors $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$ span $\mathcal{T}$, then it is easy to see that the vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell\}$ span $\mathcal{S} + \mathcal{T}$. So in the case where $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$, Problem 2(b) shows that the union of a basis for $\mathcal{S}$ and a basis for $\mathcal{T}$ is a basis for $\mathcal{S} + \mathcal{T}$. This proves that if $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$ then $\dim(\mathcal{S} + \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T})$.

More generally, we always have $\dim(\mathcal{S} + \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}) - \dim(\mathcal{S} \cap \mathcal{T})$. The easiest way to prove this is to pick a basis $\mathcal{B}$ for $\mathcal{S} \cap \mathcal{T}$, enlarge it to a basis $\mathcal{B} \cup \mathcal{B}'$ for $\mathcal{S}$, enlarge it to a basis $\mathcal{B} \cup \mathcal{B}''$ for $\mathcal{T}$, and show that $\mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}''$ is a basis for $\mathcal{S} + \mathcal{T}$.

Problem 4 was Example 4 of §2.2, on p. 101 of Penney. Of course, it is easier to solve with our present tools than it was back in §2.2.

Regarding Solution #2 to Problem 6(a): in Problem Set 12 (Exercise 8 of §4.5 of Penney), you proved both algebraically and geometrically that the projection matrix $P = A(A^tA)^{-1}A^t$ is idempotent.

In Problem 6(d), the quoted statement becomes true if we add the condition that $AB = BA$. For in this case, $A^2 = A$ and $B^2 = B$ imply $(AB)^2 = ABAB = A(BA)B = A(AB)B = A^2B^2 = AB$.

The solution to Problem 6(e) shows that *every* idempotent 2 by 2 matrix is either the identity matrix, the zero matrix, or else equals

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{ad}{ad-bc} & -\dfrac{ab}{ad-bc} \\ \dfrac{cd}{ad-bc} & -\dfrac{bc}{ad-bc} \end{bmatrix}$$

where $ad - bc \neq 0$. Throwing in the condition that a, b, c, and d are all nonzero gives the set of all possible answers to Problem 6(a).[**]

In the extra-credit problem, we obviously cannot define $T : \mathcal{C}[0, 1] \longrightarrow \mathcal{C}[0, 1]$ by $T(f) = f''$, since for an arbitrary continuous function $f : [0, 1] \longrightarrow \mathbb{R}$, it is not necessarily the case that f'' is a continuous function from $[0, 1]$ to $\mathbb{R}$. In fact, as noted at the end of Solution #2 to the extra-credit problem, f' might not even be a function from $[0, 1]$ to $\mathbb{R}$ at all: the value of f' could be nonexistent at every single point of $[0, 1]$.

Solution #2 to the extra-credit problem was based on an important algebraic property of vector spaces. If $\mathcal{V}$ is any vector space, and $\mathcal{S} \subseteq \mathcal{V}$ is any subspace, then there exists a subspace $\mathcal{T} \subseteq \mathcal{V}$ such that $\mathcal{S} + \mathcal{T} = \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$. (This property is known as *semisimplicity.*)

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Graded Exams

To view your graded final exam, go to 104 Ritter Hall between Jan. 13, 2004, and Jan. 13, 2005. The graded final exams will not be available before or after this range of dates.

As announced in the syllabus, every course grade I assign (with the exception of the special grades "I," "NR," and "X") shall be considered final once the grade is recorded; I will request a change of grade, when warranted, if a computational or procedural error occurred in the original assignment of the grade, but a grade will not be changed based upon a re-evaluation of any student's work.

[‖]For an example of such a function, see M. Spivak, *Calculus,* Second Edition (Publish or Perish, Inc., Houston, 1980), p. 475, or P. Franklin, *A Treatise on Advanced Calculus* (John Wiley & Sons, Inc., New York, 1940), pp. 146–147, Exercise 10.

[**]Can you exhibit Solution #2 as a special case of this?