ON 2-PRIMAL ORE EXTENSIONS

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Abstract. When R is a local ring with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition.

1. Background and motivation

In [13, Proposition 3.7], it was shown that if R is a local, one-sided artinian ring with an automorphism σ, then the skew polynomial ring $R[x;\sigma]$ must satisfy the (PS I) condition (defined below). The main result of this paper extends [13, Proposition 3.7] to the case of a general Ore polynomial ring $R[x;\sigma,\delta]$.

Recall that in this context, σ is a ring endomorphism of R, δ is a σ-derivation of R (i.e. a map $\delta : R \longrightarrow R$ satisfying $\delta(a+b) = \delta(a)+\delta(b)$ and $\delta(ab) = \sigma(a)\delta(b)+\delta(a)b$ for all $a, b \in R$), and multiplication of polynomials in $R[x;\sigma,\delta]$ (written with left-hand coefficients) is determined by the rule $xr = \sigma(r)x + \delta(r)$ for every $r \in R$. All of our rings will be associative with 1, and generally noncommutative.

Let us recall some definitions. A ring R is called *2-primal* if the set of nilpotent elements of the ring coincides with the prime radical $\mathrm{Nil}_*(R)$. If, more generally, the set of nilpotent elements of a ring is an ideal (not necessarily the prime radical), then the ring is called an *NI-ring*. A ring R is said to *satisfy (PS I)* if for every element $a \in R$, the factor ring $R/\mathrm{ann}_r^R(aR)$ is 2-primal. Taking $a = 1$ in this definition shows that (PS I) implies 2-primal, whence the following implication chart:

$$\text{(PS I)} \quad\Longrightarrow\quad \text{2-primal} \quad\Longrightarrow\quad \text{NI-ring} \quad\Longrightarrow\quad \text{general ring}$$

In our main result, we will find that the Ore polynomial rings $R[x;\sigma,\delta]$ under consideration will either satisfy the strongest of these four conditions, or else will satisfy only the weakest.

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Some of the motivation for studying 2-primal rings and rings satisfying (PS I) comes from the early results on these rings due to G. Shin. In [16, Theorem 3.5], Shin showed that given a ring R that satisfies (PS I) as well as a condition he called (PS II) (viz., for all $a, b, c \in R$ and $n \in \mathbb{N}$, if $aR(bc)^n = 0$ then $a(RbR)^m c^m = 0$ for some $m \in \mathbb{N}$), such a ring R will be isomorphic to a ring of global sections of a sheaf whose stalks are certain factor rings of R.

Part of the attraction of 2-primal rings (in addition to their being a common generalization of commutative rings and rings without nilpotent elements) lies in the structure of their prime ideals. Shin showed in [16, Proposition 1.11] that a ring R is 2-primal if and only if every minimal prime ideal $\mathfrak{p} \subset R$ is completely prime (i.e. $R/\mathfrak{p}$ is a domain). This useful result implies, of course, that the (PS I) condition can be characterized by the requirement that all primes minimal over the appropriate annihilator ideals be completely prime.

The 2-primal condition entails various topological conditions on the prime spectrum. For example, Shin proved that the minimal-prime spectrum of a 2-primal ring is a Hausdorff space with a basis of clopen sets ([16, Proposition 4.7]). Some years later, the 2-primal condition underwent a renascence in the work of S.-H. Sun. In [17], Sun generalized a result of G. De Marco and A. Orsatti in commutative ring theory (see [4]) based on the following topological conditions on the ideals of a ring:

(i) that the prime spectrum be a normal topological space;
(ii) that the maximal-ideal spectrum be a continuous retract of the prime spectrum;
(iii) that the maximal-ideal spectrum be Hausdorff.

Sun showed that for 2-primal rings, (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii). (See [17, Theorem 2.3] for the result, and see [17, p. 188] for the equivalence of the 2-primal condition to the "weakly symmetric" condition used in [17, Theorem 2.3].)

For more on the basic properties of 2-primal rings, see [1], [16], and [17]. Many other interesting properties are obtained in [2], [3], [8], [9], and [10].

2. Strict subclasses

For the type of Ore polynomial rings we will consider, the classes "2-primal rings" and "rings satisfying (PS I)" will coalesce into a single class, as will the classes "NI-rings" and "general rings." It is therefore worth noting that the implications among these four conditions (as given in the chart in the previous section) are irreversible.

Example 2.1. General ring $\not\Longrightarrow$ NI-ring. Consider any nonzero matrix ring $R = \mathbb{M}_n(S)$ with $n > 1$.

Example 2.2. NI-ring $\not\Longrightarrow$ 2-primal. We can use [13, Example 2.1], due originally to J. Ram ([15, Example 3.2]), as a counterexample. Let k a field, let

$$S = k\left[\{t_i\}_{i \in \mathbb{Z}}\right] \Big/ \left(\{\, t_{n_1} t_{n_2} t_{n_3} \,|\, n_3 - n_2 = n_2 - n_1 > 0\}\right),$$

and let

$$R = S[x;\sigma]$$

where σ is the k-automorphism of S satisfying $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$.

It is shown in [15, Example 3.2] that the ring R is prime with nonzero Levitzki radical. With a bit more work, we can explicitly compute the Levitzki radical L-rad(R) and the upper nilradical $\mathrm{Nil}^*(R)$, and show that R is an NI-ring.

Following the terminology of [15], let us say that an element $s \in S$ is *σ-nilpotent of bounded index* if there exists some $n \in \mathbb{N}$ with $n \geqslant 3$ such that for every $m \in \mathbb{N}$ we have

$$s\sigma^m(s)\sigma^{2m}(s)\cdots\sigma^{(n-1)m}(s) = 0.$$

Since S is commutative and σ is an automorphism of S, according to [15, Theorem 3.1] the principal right ideal $sxR \subset R$ is locally nilpotent whenever the element $s \in S$ is σ-nilpotent of bounded index. For every $i \in \mathbb{Z}$, the element t_i is clearly σ-nilpotent of bounded index. Write

$$\mathfrak{m} = \sum_{i\in\mathbb{Z}} t_i S \subset S;$$

thus, $S/\mathfrak{m} \cong k$, an isomorphism that we will denote by ψ. Let

$$\mathfrak{A} = \mathfrak{m}x \cdot R = \sum_{i\in\mathbb{Z}} t_i x R.$$

As indicated above, the ideal $\mathfrak{A} \subset R$ is locally nilpotent, hence contained in the Levitzki radical L-rad(R). Note that

$$R/\mathfrak{A} \cong S \oplus kx \oplus kx^2 \oplus kx^3 \oplus \cdots, \tag{1}$$

where in the ring on the right-hand side of Equation (1), multiplication is defined by

$$\begin{aligned} &(s,\, c_1x,\, c_2x^2,\, \ldots)(s',\, d_1x,\, d_2x^2,\, \ldots) \\ &\quad = \quad (ss',\, [\psi(s)d_1 + c_1\psi(s')]x,\, [\psi(s)d_2 + c_1d_1 + c_2\psi(s')]x^2,\, \ldots) \end{aligned} \tag{2}$$

(for all $s, s' \in S$; $c_i, d_i \in k$). It is shown in [13, Example 2.1] that S is reduced; therefore, since k is also reduced, from Equations (1) and (2) we see that the factor ring $R/\mathfrak{A}$ is reduced. So every nilpotent element of the ring R is contained in the ideal $\mathfrak{A}$, which shows that

$$\mathrm{Nil}_*(R) = (0) \subsetneqq \mathfrak{A} = \text{L-rad}(R) = \mathrm{Nil}^*(R) = \{\text{nilpotent elements of } R\}.$$

Thus, R is an NI-ring that is not 2-primal.

There are, of course, many other examples of NI-rings that are not 2-primal. To cite just one: a semiprime, local ring whose nonzero maximal ideal is nil can be found in [3, Example 3.3].

The next example, easy though it is, does seem to fill a small lacuna in the literature. It appears that every published example of a 2-primal ring to date also satisfies (PS I) (cf. [14] for further discussion and examples).

Example 2.3. 2-primal $\not\Longrightarrow$ (PS I). If T is a ring and M is a (T,T)-bimodule, then one can form the ring $R = T \oplus {}_TM_T$ with multiplication defined by

$$(t_1,\, m_1)(t_2,\, m_2) = (t_1t_2,\, t_1m_2 + m_1t_2)$$

(for all $t_1, t_2 \in T$; $m_1, m_2 \in {}_TM_T$). It is not hard to see that R will be 2-primal if and only if T is 2-primal. Now let T be any 2-primal ring with a prime ideal $\mathfrak{p}$ that is not completely prime (e.g., $T = \mathbb{H}[x]$, where $\mathbb{H}$ denotes the real quaternions, and $\mathfrak{p} = (x^2+1)T$). Define the (T,T)-bimodule $M = T/\mathfrak{p}$, and let $R = T \oplus {}_TM_T$. Since T is 2-primal, R is 2-primal; however,

$$R/\mathrm{ann}_r^R((0, 1+\mathfrak{p})R) = R/(\mathfrak{p} \oplus {}_TM_T) \cong T/\mathfrak{p},$$

which is prime but not a domain, and hence not 2-primal. Thus, R is 2-primal but does not satisfy (PS I)

3. A tractable class of Ore extensions

It was shown in [1, Proposition 2.6] that a polynomial ring (in any set of central indeterminates) over a 2-primal ring will also be 2-primal. Attempts to extend this result from polynomial rings to rings of formal power series, skew polynomial rings, and differential operator rings were explored in [10], [12], and [13]. In particular, some conditions under which a skew polynomial ring or a differential operator ring will be 2-primal can be found in [13].

When we move from these "unmixed" Ore extensions to the case of an Ore polynomial ring with an endomorphism and a derivation, we face a much greater challenge. The methods used in [13] on the "unmixed" Ore extensions do not apply to the general case. In the investigation of skew polynomial rings, the 2-primal condition was studied via Shin's minimal-prime criterion, based on an understanding of extension and contraction of primes between the base ring the the skew polynomial extension. In the investigation of differential operator rings, the 2-primal condition was studied via direct examination of nilpotent elements, based on an understanding of the structure of the prime radical of the differential polynomial extension, which was determined by M. Ferrero, K. Kishimoto, and K. Motose in [5, Corollary 2.2(2)]

In the case of a general Ore extension, neither of these two methods is available to us; there are neither extension/contraction results for prime ideals nor formulas for the prime radical. Positive results *have* been found in special cases (e.g., commutative noetherian base rings, quantized derivations, etc.). Various results on the prime radical of an Ore extension were obtained by T. Y. Lam, A. Leroy, and J. Matczuk in [11, §5]. Results on the structure of prime ideals in an Ore extension were obtained by K. R. Goodearl in [6] and by Goodearl and E. S. Letzter in [7]. For example, in [6], one finds a complete classification of the primes of an Ore extension $R[x;\sigma,\delta]$, vis-à-vis their contractions to R, in the case where R is commutative noetherian and σ is an automorphism of R. Many of Goodearl's results have interesting implications for the 2-primal condition—e.g., [6, Corollary 7.8], recounted here:

Theorem 3.1 (Goodearl). *Let R be a commutative, local, artinian ring with maximal ideal $\mathfrak{m}$. Let $\sigma : R \longrightarrow R$ be an automorphism and $\delta : R \longrightarrow R$ a σ-derivation such that*

$$\mathrm{im}(1_R - \sigma) \subseteq \mathfrak{m}$$

and no proper nonzero ideal of R is δ-invariant. Assume that $\delta\sigma = q\sigma\delta$ for some central element $q \in R$ satisfying $\sigma(q) = q$ and $\delta(q) = 0$. Then for some derivation δ' on the factor ring $R/\mathfrak{m}$, the Ore extension $R[x; \sigma, \delta]$ satisfies the equation

$$R[x; \sigma, \delta] \cong \mathbb{M}_n((R/\mathfrak{m})[x; \delta'])$$

where $n = \mathrm{length}(R_R)$.

The Ore extension $R[x; \sigma, \delta]$ in Theorem 3.1 will be 2-primal if and only if R is a field. Moreover, as a 2-primal ring, $R[x; \sigma, \delta]$ will either be a smashing success or a catastrophic failure (prefiguring our main result, Theorem 3.4). For if R is a field in Theorem 3.1, then $R[x; \sigma, \delta]$ will satisfy (PS I) (since the hypotheses will force $\sigma = 1_R$, whence $R[x; \sigma, \delta]$ will be a domain). If R is not a field, then $R[x; \sigma, \delta]$ will not be an NI-ring (since $\mathrm{length}(R_R) > 1$; cf. Example 2.1).

Preparatory to our main result, we begin with a simple but useful criterion for (PS I).

Lemma 3.2. *Let R be a ring whose prime radical contains all right zero-divisors of R. Then R satisfies (PS I).*

Proof. By hypothesis, all nilpotent elements of R are contained in $\mathrm{Nil}_*(R)$, so R is 2-primal. Pick any nonzero element $a \in R$. Since the ideal $\mathrm{ann}_r^R(aR)$ consists of right zero-divisors, $\mathrm{ann}_r^R(aR) \subseteq \mathrm{Nil}_*(R)$; hence,

$$\left(R/\mathrm{ann}_r^R(aR)\right) \Big/ \mathrm{Nil}_*\left(R/\mathrm{ann}_r^R(aR)\right) \cong R/\mathrm{Nil}_*(R),$$

which is a reduced ring, since R is 2-primal. Thus, $R/\mathrm{ann}_r^R(aR)$ is 2-primal, which shows that R satisfies (PS I). □

Given a ring R with an endomorphism σ and a σ-derivation δ, following [11] we will write f_k^m for the sum of all $\binom{m}{k}$ different m-letter words consisting of k letters σ and $m-k$ letters δ. *This f_k^m notation will be used throughout the remainder of this paper.* Multiplication in the Ore extension $R[x; \sigma, \delta]$ follows the following rule, given in [11, Lemma 4.1]:

$$x^m r = \sum_{i=0}^{m} f_i^m(r) x^i. \tag{3}$$

Moreover (with the convention that the product of an empty sequence is 1),

$$\delta(r_1 r_2 \cdots r_m) = \sum_{i=1}^{m} \sigma(r_1)\sigma(r_2)\cdots\sigma(r_{i-1})\delta(r_i) r_{i+1} r_{i+2} \cdots r_m \tag{4}$$

for all $r_1, r_2, \ldots, r_m \in R$. Equations (3) and (4) can be proved by a straightforward induction on m.

We now examine the structure of certain Ore extensions of local rings.

Lemma 3.3. *Suppose R is a local ring with an endomorphism σ and a σ-derivation δ, suppose the maximal ideal $\mathfrak{m}$ of R is nilpotent, and suppose that $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$. Put $S = R[x; \sigma, \delta]$.*

(i) *For all nonnegative integers a and b, we have $(\mathfrak{m}^a S)(\mathfrak{m}^b S) \subseteq \mathfrak{m}^{a+b} S$.*
(ii) *The ideal $\mathfrak{m}S \subset S$ is completely prime.*
(iii) *The following equation holds:*

$$\begin{aligned} \mathfrak{m}S &= \mathrm{Nil}_*(S) \\ &= \{\textit{nilpotent elements of } S\} \\ &= \{\textit{right zero-divisors of } S\}. \end{aligned}$$

Proof. Notice that $\sigma(\mathfrak{m}^j) \subseteq \mathfrak{m}^j$ for all $j \in \mathbb{N}$.

(i): Let $g \in \mathfrak{m}^a S$ and $h \in \mathfrak{m}^b S$. To prove that $gh \in \mathfrak{m}^{a+b} S$, we can assume without loss of generality that g and h are monomials: $g = \alpha x^m$ and $h = \beta x^p$ where $\alpha \in \mathfrak{m}^a$ and $\beta \in \mathfrak{m}^b$. Since $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$, Equation (4) gives $\delta(\mathfrak{m}^b) \subseteq \mathfrak{m}^b$. Consequently, $f_i^m(\beta) \in \mathfrak{m}^b$ for all i. Therefore, by Equation (3),

$$gh = \sum_{i=0}^{m} \alpha f_i^m(\beta) x^{i+p} \in \mathfrak{m}^{a+b} S,$$

as desired.

(ii): Since $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$, we know that $\mathfrak{m}S$ is an ideal of S, and there is a ring isomorphism

$$S/\mathfrak{m}S \cong (R/\mathfrak{m})[x; \sigma', \delta'], \tag{5}$$

given by

$$\left(\sum_{i=0}^{m} r_i x^i \right) + \mathfrak{m}S \mapsto \sum_{i=0}^{m} (r_i + \mathfrak{m}) x^i,$$

where the endomorphism σ' and the σ'-derivation δ' are naturally induced by σ and δ (i.e. $\sigma'(r + \mathfrak{m}) = \sigma(r) + \mathfrak{m}$ and $\delta'(r + \mathfrak{m}) = \delta(r) + \mathfrak{m}$). Note that σ' is injective (even if σ is not), since σ takes units to units. Because the factor ring $R/\mathfrak{m}$ is a domain and the endomorphism σ' is injective, Equation (5) implies that the factor ring $S/\mathfrak{m}S$ is a domain, i.e. that the ideal $\mathfrak{m}S \subset S$ is completely prime.

(iii): Since (by hypothesis) the ideal $\mathfrak{m} \subset R$ is nilpotent, it follows from part (i) of the lemma that the ideal $\mathfrak{m}S \subset S$ is nilpotent, hence contained in the prime radical of S. We therefore have

$$\mathfrak{m}S \subseteq \mathrm{Nil}_*(S) \subseteq \{\text{nilpotent elements of } S\} \subseteq \{\text{right zero-divisors of } S\},$$

so it remains only to show that every right zero-divisor of S is contained in $\mathfrak{m}S$.

Assume that

$$h = \sum_{i=0}^{s} b_i x^i \notin \mathfrak{m}S,$$

and fix an arbitrary nonzero element $g \in S$. We will conclude the proof by showing that $gh \neq 0$. Let n be the largest integer for which $b_n \notin \mathfrak{m}$. Write

$$g = \sum_{i=0}^{t} a_i x^i,$$

let k be a nonnegative integer such that

$$g \in \mathfrak{m}^k S \setminus \mathfrak{m}^{k+1} S,$$

and let m be the largest integer for which $a_m \notin \mathfrak{m}^{k+1}$.

Every coefficient of the product

$$\left(\sum_{i=m+1}^{t} a_i x^i \right) h$$

is contained in $\mathfrak{m}^{k+1}$, and every coefficient of the product

$$\left(\sum_{i=0}^{m} a_i x^i \right) \left(\sum_{i=n+1}^{s} b_i x^i \right)$$

is contained in $\mathfrak{m}^{k+1}$, by part (i) of the lemma. But the highest coefficient of the product

$$\left(\sum_{i=0}^{m} a_i x^i \right) \left(\sum_{i=0}^{n} b_i x^i \right)$$

is $a_m \sigma^m(b_n) \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$. Thus $gh \notin \mathfrak{m}^{k+1} S$, and in particular, $gh \neq 0$. □

We can now prove our main theorem on the 2-primal condition in Ore extensions.

Theorem 3.4 sharpens [13, Proposition 3.7] in four respects. First, we no longer require σ to be an automorphism. (The proof of [13, Proposition 3.7], incidentally, depends crucially upon the injectivity of σ, as do many of the results on Ore extensions in [6], [7], and [11].) Second, the Ore polynomial ring is now equipped with a derivation as well as an endomorphism. Third, our condition under which the Ore extension inherits the 2-primal condition is shown to be necessary, not just sufficient. Fourth, we no longer assume that the local base ring be one-sided artinian, merely that its maximal ideal be nilpotent.

This theorem, it will be noted, recovers our observations in the paragraph following Theorem 3.1 about the Ore extensions $R[x; \sigma, \delta]$ in that theorem.

Theorem 3.4. *Suppose R is a local ring with an endomorphism σ and a σ-derivation δ, and suppose the maximal ideal $\mathfrak{m}$ of R is nilpotent.*

(i) *If $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$, then the Ore extension $R[x; \sigma, \delta]$ satisfies (PS I) (in particular, $R[x; \sigma, \delta]$ is 2-primal).*

(ii) *If $\delta(\mathfrak{m}) \not\subseteq \mathfrak{m}$, then the Ore extension $R[x; \sigma, \delta]$ is not an NI-ring (in particular, $R[x; \sigma, \delta]$ is not 2-primal).*

Proof. (i): Apply Lemma 3.3(iii) and Lemma 3.2.

(ii): Suppose $\delta(\mathfrak{m}) \not\subseteq \mathfrak{m}$. Choose $r \in \mathfrak{m}$ such that $\delta(r)$ is a unit. The element $r \in R[x; \sigma, \delta]$ is of course nilpotent. We will show that the element $xr \in R[x; \sigma, \delta]$ is not nilpotent, completing the proof.

Claim. For every nonnegative integer n,

$$(xr)^{2^n} = [\delta(r)]^{2^n} + \sum_{k=0}^{2^n} a_k x^k$$

where $a_k \in \mathfrak{m}$ for all k.

To prove the claim, we induct on n. As in the proof of Lemma 3.3, let us observe that $\sigma(\mathfrak{m}^j) \subseteq \mathfrak{m}^j$ for all $j \in \mathbb{N}$.

The $n = 0$ case of the claim is simply $xr = \sigma(r)x + \delta(r)$. Assuming the claim true for n, we find that

$$(xr)^{2^{n+1}} = \left([\delta(r)]^{2^n} + \sum_{k=0}^{2^n} a_k x^k\right)^2 = [\delta(r)]^{2^{n+1}} + g_1 + g_2 + g_3,$$

where the elements

$$g_1 = [\delta(r)]^{2^n}\left(\sum_{k=0}^{2^n} a_k x^k\right), \qquad g_2 = \left(\sum_{k=0}^{2^n} a_k x^k\right)[\delta(r)]^{2^n},$$

and

$$g_3 = \left(\sum_{k=0}^{2^n} a_k x^k\right)\left(\sum_{k=0}^{2^n} a_k x^k\right)$$

all lie in the right ideal $\mathfrak{m}S$, using the inductive hypothesis. This completes the induction, proving the claim.

By the claim, for all $n \in \mathbb{N}$, the constant coefficient of $(xr)^{2^n}$ is a unit, hence $(xr)^{2^n} \neq 0$. Thus, $xr \in R[x; \sigma, \delta]$ is not nilpotent. □

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