Putnam Mathematical Competition

The Putnam Competition is the premier national undergraduate mathematics contest, which will next be held on Saturday, December 7, 2024. Approximately 3,900 undergraduate students from 470 colleges and universities throughout the U.S. and Canada are expected to compete. Registration is free. In addition to awarding cash prizes (up to $3,500) to the 25 highest scoring participants, the Mathematical Association of America will publish the names and schools of the top 500 students. The national directors of the Putnam Competition make a point of commending these students to the attention of graduate school admissions committees. (There is no risk in attempting the Putnam: participants scoring below the top 500 are not publicly identified.)

In 2023 a score of 28/120 would have put you in the top 500. The exam, needless to say, is not easy! *If you can solve even one problem completely correctly, you will be making a significant contribution to the SLU team.* A good performance can bring great prestige to both you and the university. The Putnam provides an opportunity for the creative application to novel problems of important mathematical techniques and ideas spanning much of the undergraduate curriculum.

Interested students should contact the local supervisor, Greg Marks (**marks@gmarks.org**), and might consider enrolling in Professor Marks's one-unit S/U course Math 2690, Mathematical Problem Solving.

History. The competition began in 1938 and is designed to stimulate a healthful rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the Harvard Graduates' Magazine in which he described the merits of an intellectual intercollegiate competition.

Description. The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as "survey of the foundations of mathematics." It is also expected that the self-contained questions involving elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be entirely foreign to the contestant's experience.

Sample Exam
The Eighty-Fourth William Lowell Putnam Mathematical Competition
Saturday, December 2, 2023

Problems A1 through A6 were given during the 3-hour morning session of the exam; problems B1 through B6 were given during the 3-hour afternoon session.

A1. For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

A2. Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \le |k| \le n$. Find all other real numbers x for which $p(1/x) = x^2$.

A3. Determine the smallest positive real number r such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \le |g(x)|$ for all x,

(d) $|g'(x)| \le |f(x)|$ for all x, and

(e) $f(r) = 0$.

A4. Let $v_1, \dots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \dots, a_{12}$ such that $\|a_1v_1 + \cdots + a_{12}v_{12} - v\| < \varepsilon$.

A5. For a nonnegative integer k, let $f(k)$ be the number of ones in the base 3 representation of k. Find all complex numbers z such that

$$\sum_{k=0}^{3^{1010}-1} (-2)^{f(k)}(z+k)^{2023} \;=\; 0.$$

A6. Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even." On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon$ the number k was chosen on the kth turn$\}$ matches his goal. For which values of n does Bob have a winning strategy?

B1. Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \le i \le m$ and $1 \le j \le n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \le i \le m-1$ and $1 \le j \le n-1$. If a coin occupies the square (i,j) with $i \le m-1$ and $j \le n-1$ and the squares $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

B2. For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

B3. A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_0 - y_1,\ y_3 - y_2,\ \dots,\ y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \ge 2$.

B4. For a nonnegative integer n and a strictly increasing sequence of real numbers $t_0, t_1, \dots, t_n$, let $f(t)$ be the corresponding real-valued function defined for $t \ge t_0$ by the following properties:

(a) $f(t)$ is continuous for $t \ge t_0$, and is twice differentiable for all $t > t_0$ other than $t_1, \dots, t_n$;

(b) $f(t_0) = 1/2$;

(c) $\lim_{t \to t_k^+} f'(t) = 0$ for $0 \le k \le n$;

(d) For $0 \le k \le n-1$, we have $f''(t) = k+1$ when $t_k < t < t_{k+1}$, and $f''(t) = n+1$ when $t > t_n$.

Considering all choices of n and $t_0, t_1, \dots, t_n$ such that $t_k \ge t_{k-1} + 1$ for $1 \le k \le n$, what is the least possible value of T for which $f(t_0 + T) = 2023$?

B5. Determine which positive integers n have the following property: For all integers m that are relatively prime to n, there exists a permutation $\pi\colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $\pi(\pi(k)) \equiv mk \bmod n$ for all $k \in \{1, 2, \dots, n\}$.

B6. Let n be a positive integer. For i and j in $\{1, 2, \dots, n\}$, let $s(i,j)$ be the number of pairs (a,b) of nonnegative integers satisfying $ai + bj = n$. Let S be the n-by-n matrix whose (i,j)-entry is $s(i,j)$. For example, when $n = 5$, we have

$$S \;=\; \begin{bmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the determinant of S.