Putnam Mathematical Competition

The Putnam Competition is the premier national undergraduate mathematics contest, which will next be held on Saturday, December 7, 2019.* Approximately 4,600 undergraduate students from 570 colleges and universities throughout the U.S. and Canada are expected to compete. Registration is free. In addition to awarding cash prizes (up to $3,500) to the 25 highest scoring participants, the Mathematical Association of America will publish the names and schools of the top 500 students. The national directors of the Putnam Competition make a point of commending these students to the attention of graduate school admissions committees. (There is no risk in attempting the Putnam: participants scoring below the top 500 are not publicly identified.)

In 2018 a score of 23/120 would have put you in the top 500. The exam, needless to say, is not easy! *If you can solve even one problem completely correctly, you will be making a significant contribution to the SLU team.* A good performance can bring great prestige to both you and the university. The Putnam provides an opportunity for the creative application to novel problems of important mathematical techniques and ideas spanning much of the undergraduate curriculum.

Interested students should contact the local supervisor, Greg Marks (`marks@slu.edu`), and might consider enrolling in Professor Marks's one-unit S/U course Math 2690, Mathematical Problem Solving.

History. The competition began in 1938 and is designed to stimulate a healthful rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the Harvard Graduates' Magazine in which he described the merits of an intellectual intercollegiate competition.

Description. The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as "survey of the foundations of mathematics." It is also expected that the self-contained questions involving elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be entirely foreign to the contestant's experience.

Sample Exam
The Seventy-Ninth William Lowell Putnam Mathematical Competition
Saturday, December 1, 2018

Problems A1 through A6 were given during the 3-hour morning session of the exam; problems B1 through B6 were given during the 3-hour afternoon session.

A1. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

A2. Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n-1)\times(2^n-1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \varnothing; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

*The registration deadline is October 1, 2019.

A3. Determine the greatest possible value of

$$\sum_{i=1}^{10} \cos(3x_i)$$

for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying

$$\sum_{i=1}^{10} \cos(x_i) = 0.$$

A4. Let m and n be positive integers with $\gcd(m,n) = 1$, and let

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1}gh^{a_2}\cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

A5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

A6. Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

B1. Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \;\middle|\; 0 \leq a \leq 2, \quad 0 \leq b \leq 100, \quad \text{and} \quad a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

B2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C}\colon\ |z| \leq 1\}$.

B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

B4. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

B5. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

B6. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$