Putnam Mathematical Competition

The Putnam Competition is the premier national undergraduate mathematics contest, which will next be held on Saturday, December 1, 2018.* Approximately 4,600 undergraduate students from 570 colleges and universities throughout the U.S. and Canada are expected to compete. Registration is free. In addition to awarding cash prizes (up to $3,500) to the 25 highest scoring participants, the Mathematical Association of America will publish the names and schools of the top 500 students. The national directors of the Putnam Competition make a point of commending these students to the attention of graduate school admissions committees. (There is no risk in attempting the Putnam: participants scoring below the top 500 are not publicly identified.)

In 2017 a score of 26/120 would have put you in the top 500. The exam, needless to say, is not easy! *If you can solve even one problem completely correctly, you will be making a significant contribution to the SLU team.* A good performance can bring great prestige to both you and the university. The Putnam provides an opportunity for the creative application to novel problems of important mathematical techniques and ideas spanning much of the undergraduate curriculum.

Interested students should contact the local supervisor, Greg Marks (**marks@slu.edu**), and might consider enrolling in Professor Marks's one-unit S/U course Math 2690, Mathematical Problem Solving.

History. The competition began in 1938 and is designed to stimulate a healthful rivalry in mathematical studies in the colleges and universities of the United States and Canada. It exists because Mr. William Lowell Putnam had a profound conviction in the value of organized team competition in regular college studies. Mr. Putnam, a member of the Harvard class of 1882, wrote an article for the December 1921 issue of the Harvard Graduates' Magazine in which he described the merits of an intellectual intercollegiate competition.

Description. The examination will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is presumed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included that cut across the bounds of various disciplines, and self-contained questions that do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as "survey of the foundations of mathematics."

Sample Exam
The Seventy-Eighth William Lowell Putnam Mathematical Competition
Saturday, December 2, 2017

Problems A1 through A6 were given during the 3-hour morning session of the exam; problems B1 through B6 were given during the 3-hour afternoon session.

A1. Let S be the smallest set of positive integers such that

a) 2 is in S,

b) n is in S whenever n^2 is in S, and

c) $(n+5)^2$ is in S whenever n is in S.

Which positive integers are not in S?
(The set S is "smallest" in the sense that S is contained in any other such set.)

A2. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

*The registration deadline is October 1, 2018.

A3. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n \;=\; \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

A4. A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

A5. Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A, B, and C, take turns in the order $A, B, C, A, \ldots$ choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

A6. The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

B1. Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda\,\overrightarrow{PA_1}$.

B2. Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N \;=\; a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

B3. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

B4. Evaluate the sum

$$\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)$$
$$= 3\cdot\frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3\cdot\frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3\cdot\frac{\ln 10}{10} - \cdots.$$

(As usual, $\ln x$ denotes the natural logarithm of x.)

B5. A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

B6. Find the number of ordered 64-tuples $(x_0, x_1, \ldots, x_{63})$ such that $x_0, x_1, \ldots, x_{63}$ are distinct elements of $\{1, 2, \ldots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

is divisible by 2017.